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Study of a Four-Server Feedback Queuing Model with Finite Chances of Customer Returns to Any Server

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Abstract:
A queuing model has been created for a system with four servers, allowing customers to return to any server. Each customer can revisit upto finie number of times and may need services from one or all servers. If a customer requires assistance from multiple servers, they will first go to the first server and then can choose to visit either of the other three. After receiving service from any server, the customer can either return to another server or exit the system based on their satisfaction. The likelihood of a customer leaving a server changes with each visit, differing from their previous departure probability. The Mean Queue Length of the system is calculated from the steady state equations developed from the model.

Keywords: Feedback, Queuing System, Poisson Process, Four Server, Mean Queue Length of the system.

1. Introduction

In the Queueing theory we deal with the study of optimizing the waiting time of service. Generally, queues are formed when the supply of service does not balance with the demand of service. We face queueing problem everywhere in our day-to-day life such as in the banks, post offices, hospitals, airports, highway tolls etc. Many authors did a lot of work on queueing theory.

Mean Queue length of a feedback queueing system having three service channels wherein a customer may go forward/back to any service channel can be found. But there is no restriction on the number of such movements [1]. Mean queue length of the feedback queueing system changes with respect to different queueing characteristics keeping the other parameters constant which comprises of three servers linked in series hierarchically [2]. The sensitivity analysis of a multiple-server queuing system with reverse balking and reneging is carried out by obtaining the performance measures like average system size, average rate of reneging, and average rate of reverse balking are obtained [3]. Designing the queue size in terms of the steady-state probabilities is a more reasonable and rational method than the expected queue length [4]. A simple and direct approach have been provided to determine quality control policy and optimization of the Markovian feedback customers that has single-server and limited system capacity under steady-state situation adding the concepts of balking and retention of reneged customers [5]. After the completion of servicing in the high-speed server of a system with heterogeneous servers, MMPP flow, and instant feedback, primary calls according to the Bernoulli scheme either leave the system or immediately require re-servicing. Repeated (feedback) calls are served in a low-speed server, and after the service is completed, feedback calls can be repeated many times [6]. System size along with other performance measures of a multiserver Markovian queuing model can be obtained under the assumption that customers are state aware so that their impatience is state dependent [7]. A stochastic queuing model developed and solved for the system facing issues on the impatient behaviour of customers can help firms measure their performance well in advance and device strategies for effective management of the system [8]. Transient state probabilities for exact number of arrivals and departures from the two-state retrial queueing systems applicable in computer systems with feedback having two identical parallel servers, can be obtained when both, one or none of the servers is busy [9]. Mean queue length and other queueuing characteristics of a system of complex feedback queue model containing three subsystems; one is comprised with two biserial service channels and other with two parallel service channels can be obtained in which both the subsystems are commonly connected with a central service channel [10].An extensive review of studies addressing queueing-related problems explicitly related to patient scheduling and queueing in emergency departments is provided and discussed scientific methodologies used to analyse and optimize algorithms, priority models, queueing models, simulation, and statistical approaches [11]. An M/M/1 retrial queue under differentiated vacations and Bernoulli feedback policy is analysed in which on receiving the service, if the customer is not satisfied, then he may join the retrial group again with some probability and demand for service or may leave the system with the complementary probability [12]. Some performance measures such as mean queue length of the system, waiting time etc. of a single server batch service queue model with feedback and second optional service under a transient and steady state environment can be analyzed [13]. Mean queue length of a four-server hierarchical structured queuing system with feedback and revisit of customer at most once to any of the servers increases and decreases with respect to different parameters [14].

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No author worked on four server feedback queueing system with the facility of having revisits of customer upto finite number of times. Keeping this in view we have calculated the mean queue lengths of the system using the steady state equations and generating function technique.

2. Notation

λ: Mean Arrival rate at 1st server (S_1)
μ_1: service rate of 1st server (S_1)
μ_2: service rate of 2ndserver (S_2)
μ_3: service rate of 3rd server (S_3)
μ_4: service rate of 4th server (S_4)
p^i_{12}: the probability of customer going from 1stto 2nd server ith time.
p^i_{13}: the probability of customer going from 1stto 3rd server ith time.
p^i_{14}: the probability of customer going from 1stto 4th server ith time.
p^i_2: the probability of exit of customer from 2nd server ith time.
p^i_{23}: the probability of customer going from 2nd to 3rd server ith time.
p^i_{24}: the probability of customer going from 2nd to 4th server ith time.
p^i_{21}: the probability of customer going from 2nd to 1st server ith time.
p^i_3: the probability of exit of customer from 3rd server ith time.
p^i_{31}: the probability of customer going from 3rd to 1st server ith time.
p^i_{32}: the probability of customer going from 3rd to 2nd server ith time.
p^i_{34}: the probability of customer going from 3rd to 4th server ith time
p^i_4: the probability of exit of customer from 4th server ith time.
p^i_{41} : the probability of exit of customer from 4th to 1st server ith time.
p^i_{42}: the probability of customer going from 4th to 2nd server ith time.
p^i_{43}: the probability of customer going from 4th to 3rd server ith time.
$A_{12} = \sum_{i=1}^{n} a^i p^i_{12}$ $A_{13} = \sum_{i=1}^{n} a^i p^i_{13}$

$$A_{14} = \sum_{i=1}^{n} a^i p^i_{14}$$

$$B_2 = \sum_{i=1}^{n} b^i p^i_2 ,\ B_{21} = \sum_{i=1}^{n-1} b^i p^i_{21} ,\ B_{23} = \sum_{i=1}^{n} b^i p^i_{23} ,\ B_{24} = \sum_{i=1}^{n} b^i p^i_{24}$$

$$C_3 = \sum_{i=1}^{n} c^i p^i_3 \quad ,\ C_{34} = \sum_{i=1}^{n} c^i p^i_{34} ,\ C_{31} = \sum_{i=1}^{n-1} c^i p^i_{31} ,\ C_{32} = \sum_{i=1}^{n-1} c^i p^i_{32}$$

$$D_4 = \sum_{i=1}^{n} d^i p^i_4 ,\ D_{43} = \sum_{i=1}^{n-1} d^i p^i_{43} ,\ D_{42} = \sum_{i=1}^{n-1} d^i p^i_{42} ,\ D_{41} = \sum_{i=1}^{n-1} d^i p^i_{41}$$

3. Formulation of Problem

The queue network consists of four service channels in such a manner that first server (S_1) is centrally linked with the remaining three parallel servers (S_2), (S_3) and (S_4). It is assumed that customer arrives at first server (S_1) from outside the system and then may go to any one of the second (S_2), third (S_3) and fourth (S_4) server. The situation has been shown by the following sate transition diagram:

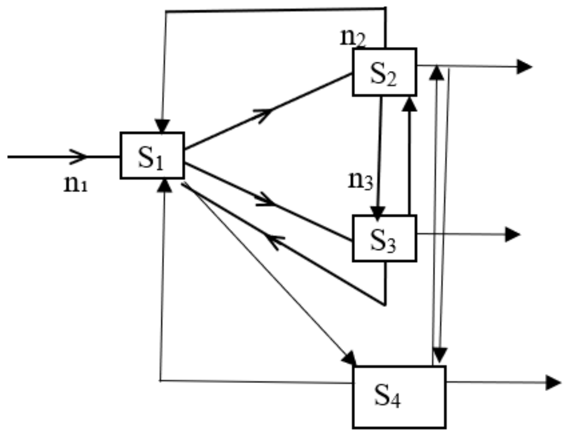

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Diagram Showing Movement of the Customers from Various Servers

A customer either goes to the second, third, or fourth server after receiving service from the first server ith time, so that $p^i_{12} + p^i_{13} + p^i_{14} = 1$. Once a customer is satisfied, they can leave the system from the second server or proceed to the third, or fourth server or back to first server so that $p^i_2 + p^i_{21} + p^i_{23} + p^i_{24} = 1$. In order to $p^i_3 + p^i_{31} + p^i_{32} + p^i_{34} = 1$, s/he can leave the system from the third server or go to the fourth, second, or first server. In a similar vein, the user may leave the system from the fourth server or switch to the third, second, or first server. Thus, $p^i_4 + p^i_{41} + p^i_{42} + p^i_{43} = 1$.

Let Q_{n_1,n_2,n_3,n_4} is the probability of having n_1, n_2, n_3, n_4 customers at server 1st, 2nd, 3rd and 4th server at any time t.

The steady–state equations for different values of n_1, n_2, n_3, and n_4 are given by:

$$
\begin{aligned}
(\lambda + \mu_1 + \mu_2 + \mu_3 + \mu_4)\, Q_{n_1,n_2,n_3,n_4} &= \lambda Q_{n_1-1,n_2,n_3,n_4} + \\
&+ \mu_1 A_{12}\, Q_{n_1+1,n_2-1,n_3,n_4} + \mu_1\, A_{14}\, Q_{n_1+1,n_2,n_3,n_4-1} + \\
&+ \mu_1\, A_{13} Q_{n_1+1,n_2,n_3-1,n_4} + \mu_2\, B_{21}\, Q_{n_1-1,n_2+1,n_3,n_4} \\
&+ \mu_2 B_2\, Q_{n_1,n_2+1,n_3,n_4} + \mu_2\, B_{23}\, Q_{n_1,n_2+1,n_3-1,n_4} \\
&+ \mu_2\, B_{24}\, Q_{n_1,n_2+1,n_3,n_4-1} + \mu_3\, C_3\, Q_{n_1,n_2,n_3+1,n_4} \\
&+ \mu_3\, C_{31}\, Q_{n_1-1,n_2,n_3+1,n_4} + \mu_3\, C_{32}\, Q_{n_1,n_2-1,n_3+1,n_4} \\
&+ \mu_3\, C_{34} Q_{n_1,n_2,n_3+1,n_4-1} + \mu_4\, D_4\, Q_{n_1,n_2,n_3,n_4+1} \\
&+ \mu_4 D_{41}\, Q_{n_1-1,n_2,n_3,n_4+1} + \mu_4\, D_{42}\, Q_{n_1,n_2-1,n_3,n_4+1} \\
&+ \mu_4\, D_{43}\, Q_{n_1,n_2,n_3-1,n_4+1} \qquad \ldots(1)
\end{aligned}
$$

Put $n_1 = 0$

$$
\begin{aligned}
(\lambda + \mu_2 + \mu_3 + \mu_4)\, Q_{0,n_2,n_3,n_4} &= \mu_1\, Q_{12} Q_{1,n_2-1,n_3,n_4} \\
&+ \mu_1 A_{13}\, Q_{1,n_2,n_3-1,n_4} + \mu_1\, A_{14} Q_{1,n_2,n_3,n_4-1} \\
&+ \mu_2 B_2\, Q_{0,n_2+1,n_3,n_4} + \mu_2\, B_{23}\, Q_{0,n_2+1,n_3-1,n_4} \\
&+ \mu_2 B_{24} Q_{0,n_2+1,n_3,n_4-1} + \mu_3\, C_3\, Q_{0,n_2,n_3+1,n_4} \\
&+ \mu_3 C_{32}\, Q_{0,n_2-1,n_3+1,n_4} + \mu_3\, C_{34}\, Q_{0,n_2,n_3+1,n_4-1} \\
&+ \mu_1 D_4\, Q_{0,n_2,n_3,n_4+1} + \mu_4\, D_{42}\, Q_{0,n_2-1,n_3,n_4+1} \\
&+ \mu_4 D_{43}\, Q_{0,n_2,n_3-1,n_4+1} \qquad \ldots(2)
\end{aligned}
$$

Put $n_2 = 0$

$$
\begin{aligned}
(\lambda + \mu_1 + \mu_3 + \mu_4) Q_{n_1,0,n_3,n_4} &= \lambda Q_{n_1-1,0,n_3,n_4} \\
&+ \mu_1 A_{14}\, Q_{n_1+1,0,n_3,n_4-1} + \mu_1 A_{13}\, Q_{n_1+1,0,n_3-1,n_4} \\
&+ \mu_2\, B_{21} Q_{n_1-1,1,n_3,n_4} + \mu_2\, B_2\, Q_{n_1,1,n_3,n_4} \\
&+ \mu_2\, B_{23}\, Q_{n_1,1,n_3-1,n_4} + \mu_2\, B_{24}\, Q_{n_1,1,n_3,n_4-1} \\
&+ \mu_3\, C_3\, Q_{n_1,0,n_3+1,n_4} + \mu_3\, C_{31}\, Q_{n_1-1,0,n_3+1,n_4} \\
&+ \mu_3\, C_{34}\, Q_{n_1,0,n_3+1,n_4-1} + \mu_4\, D_4\, Q_{n_1,0,n_3,n_4+1} \\
&+ \mu_4\, D_{41}\, Q_{n_1-1,0,n_3,n_4+1} + \mu_4 D_{43} Q_{n_1,0,n_3-1,n_4+1} \qquad \ldots(3)
\end{aligned}
$$

Put $n_3 = 0$

$$
\begin{aligned}
(\lambda + \mu_1 + \mu_2 + \mu_4) Q_{n_1,n_2,0,n_4} &= \lambda\, Q_{n_1-1,n_2,0,n_4} \\
&+ \mu_1 A_{12} Q_{n_1+1,n_2-1,0,n_4} + \mu_1\, A_{14}\, Q_{n_1+1,n_2,0,n_4-1} \\
&+ \mu_2 B_{21}\, Q_{n_1-1,n_2+1,0,n_4} + \mu_2 B_2\, Q_{n_1,n_2+1,0,n_4} \\
&+ \mu_2\, B_{24} Q_{n_1,n_2+1,0,n_4-1} + \mu_3\, C_3\, Q_{n_1,n_2,1,n_4} \\
&+ \mu_3 C_{31}\, Q_{n_1-1,n_2,1,n_4} + \mu_3\, C_{32}\, Q_{n_1,n_2-1,1,n_4} \\
&+ \mu_3 C_{34}\, Q_{n_1,n_2,1,n_4-1} + D_4\, Q_{n_1,n_2,0,n_4+1}
\end{aligned}
$$

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$$+ \mu_4 D_{41} Q_{n_1-1,n_2,0,n_4+1} + \mu_4 D_{42} Q_{n_1,n_2-1,0,n_4+1} \quad \ldots(4)$$

Put $n_4 = 0$

$$(\lambda + \mu_1 + \mu_2 + \mu_3) Q_{n_1,n_2,n_3,0} = \lambda Q_{n_1-1,n_2,n_3,0}$$
$$+ \mu_1 A_{12} Q_{n_1+1,n_2-1,n_3,0} + \mu_1 A_{13} Q_{n_1+1,n_2,n_3-1,0}$$
$$+ \mu_2 B_{21} Q_{n_1-1,n_2+1,n_3,0} + \mu_2 B_2 Q_{n_1,n_2+1,n_3,0}$$
$$+ \mu_2 B_{23} Q_{n_1,n_2+1,n_3-1,0} + \mu_3 C_3 Q_{n_1,n_2,n_3+1,0}$$
$$+ \mu_3 C_{31} Q_{n_1-1,n_2,n_3+1,0} + \mu_3 C_{32} Q_{n_1,n_2-1,n_3+1,0}$$
$$+ \mu_4 D_4 Q_{n_1,n_2,n_3,1} + \mu_4 D_{41} Q_{n_1-1,n_2,n_3,1}$$
$$+ \mu_4 D_{42} Q_{n_1,n_2-1,n_3,1} + \mu_4 D_{43} Q_{n_1,n_2,n_3-1,1} \quad \ldots(5)$$

Put $n_4 = n_2 = 0$

$$(\lambda + \mu_3 + \mu_4) Q_{0,0,n_3,n_4} = \mu_1 A_{14} Q_{1,0,n_3,n_4-1}$$
$$+ \mu_1 A_{13} Q_{1,0,n_3-1,n_4} + \mu_2 B_2 Q_{0,1,n_3,n_4}$$
$$+ \mu_2 B_{23} Q_{0,1,n_3-1,n_4} + \mu_2 B_{24} Q_{0,1,n_3,n_4-1}$$
$$+ \mu_3 C_3 Q_{0,0,n_3+1,n_4} + \mu_3 C_{34} Q_{0,0,n_3+1,n_4-1}$$
$$+ \mu_4 D_4 Q_{0,0,n_3,n_4+1} + \mu_4 D_{43} Q_{0,0,n_3-1,n_4+1} \quad \ldots(6)$$

Put n_1, $n_3 = 0$

$$(\lambda + \mu_2 + \mu_4) Q_{0,n_2,0,n_4} = \mu_1 A_{12} Q_{1,n_2-1,0,n_4}$$
$$\mu_1 A_{14} Q_{1,n_2,0,n_4-1} + \mu_2 B_2 Q_{0,n_2+1,0,n_4}$$
$$+ \mu_2 B_{24} Q_{0,n_2+1,0,n_4-1} + \mu_3 C_3 Q_{0,n_2,1,n_4}$$
$$+ \mu_3 C_{32} Q_{0,n_2-1,1,n_4} + \mu_3 C_{34} Q_{0,n_2,1,n_4-1}$$
$$+ \mu_4 D_4 Q_{0,n_2,0,n_4+1}$$
$$+ \mu_4 D_{42} Q_{0,n_2-1,0,n_4+1} \quad \ldots(7)$$

Put n_1, $n_4 = 0$

$$(\lambda + \mu_2 + \mu_3) Q_{0,n_2,n_3,0} = \mu_1 A_{12} Q_{1,n_2-1,n_3,0}$$
$$+ \mu_1 A_{13} Q_{1,n_2,n_3-1,0} + \mu_2 B_2 Q_{0,n_2+1,n_3,0}$$
$$+ \mu_2 B_{23} Q_{0,n_2+1,n_3-1,0} + \mu_3 C_3 Q_{0,n_2,n_3+1,0}$$
$$+ \mu_3 C_{32} Q_{0,n_2-1,n_3+1,0} + \mu_4 D_4 Q_{0,n_2,n_3,1}$$
$$+ \mu_4 D_{42} Q_{0,n_2-1,n_3,1} + \mu_4 D_{43} Q_{0,n_2,n_3-1,1} \quad \ldots(8)$$

Put n_2 , $n_3 = 0$

$$(\lambda + \mu_1 + \mu_4) Q_{n_1,0,0,n_4} = \lambda Q_{n_1-1,0,0,n_4}$$
$$+ \mu_1 Q_{14} Q_{n_1+1,0,0,n_4-1} + \mu_2 B_2 , Q_{n_1-1,1,0,n_4}$$
$$+ \mu_2 B_2 Q_{n_1,1,0,n_4} + \mu_2 B_{24} Q_{n_1,1,0,n_4-1}$$
$$+ \mu_3 C_3 Q_{n_1,0,0,1n_4} + \mu_3 C_{31} Q_{n_1-1,0,1,n_4}$$
$$+ \mu_3 C_{34} Q_{n_1,0,1,n_4-1} + \mu_4 D_4 Q_{n_1,0,0,n_4+1}$$
$$+ \mu_4 D_{41} Q_{n_1-1,0,0,n_4+1} \quad \ldots(9)$$

Put n_2, $n_4 = 0$

$$(\lambda + \mu_1 + \mu_3) Q_{n_1,0,n_3,0} = \lambda Q_{n_1-1,0,n_3,0}$$
$$+ \mu_1 A_{13} Q_{n_1+1,0,n_3-1,0} + \mu_2 B_{21} Q_{n_1-1,1,n_3,0}$$

$$+\mu_2 B_2 Q_{n_1,1,n_3,0} + \mu_2 B_{23} Q_{n_1,1,n_3-1,0}$$
$$+\mu_3 C_3 Q_{n_1,0,n_3+1,0} + \mu_3 C_{31} Q_{n_1-1,0,n_3+1,0}$$
$$+\mu_4 D_4 Q_{n_1,0,n_3,1} + \mu_4 D_{41} Q_{n_1-1,0,n_3,1}$$
$$+\mu_4 D_{43} Q_{n_1,0,n_3-1,1} \quad \text{….(10)}$$

For n_3, $n_4 = 0$

$$(\lambda+\mu_1+\mu_2) Q_{n_1,n_2,0,0} = \lambda Q_{n_1-1,n_2,0,0} +$$
$$+\mu_1 Q_{12} Q_{n_1+1,n_2-1,0,0} + \mu_2 B_{21} Q_{n_1-1,n_2+1,0,0}$$
$$+\mu_2 B_2 Q_{n_1,n_2+1,0,0} + \mu_3 C_3 Q_{n_1,n_2,1,0}$$
$$+\mu_3 C_{31} Q_{n_1-1,n_2,1,0} + \mu_3 C_{32} Q_{n_1,n_2-1,1,0} \quad \text{…(11)}$$

For n_1, n_2, $n_3 = 0$

$$(\lambda+\mu_4) Q_{0,0,0,n_4} = \mu_1 A_{14} Q_{1,0,0,n_4-1}$$
$$+\mu_2 B_2 Q_{0,1,0,n_4} + \mu_2 B_{24} Q_{0,1,0,n_4-1}$$
$$+\mu_3 C_3 Q_{0,0,1,n_4} + \mu_3 C_{34} Q_{0,0,1,n_4-1}$$
$$+\mu_4 D_4 Q_{0,0,0,n_4+1} \quad \text{…(12)}$$

For n_1, n_3, $n_4 = 0$

$$(\lambda+\mu_2) Q_{0,n_2,0,0} = \mu_1 A_{12} Q_{1,n_2-1,0,0}$$
$$\mu_2 B_2 Q_{0,n_2+1,0,0} + \mu_3 C_3 Q_{0,n_2,1,0}$$
$$+\mu_3 C_{32} Q_{0,n_2-1,1,0} + \mu_4 D_4 Q_{0,n_2,0,1}$$
$$+\mu_4 D_{42} Q_{0,n_2-1,0,1} \quad \text{…(13)}$$

For n_2, n_3, $n_4 = 0$

$$(\lambda+\mu_1) Q_{n_1,0,0,0} = \lambda Q_{n_1-1,0,0,0} + \mu_2 B_{21} Q_{n_1-1,1,0,0}$$
$$+\mu_2 B_2 Q_{n_1,1,0,0} + \mu_3 C_3 Q_{n_1,0,1,0}$$
$$+\mu_3 C_{31} Q_{n_1-1,0,1,0} + \mu_4 D_4 Q_{n_1,0,0,1}$$
$$+\mu_4 D_{41} Q_{n_1-1,0,0,1} \quad \text{…(14)}$$

For $n_1 = n_2 = n_4 = 0$

$$(\lambda+\mu_3) Q_{0,0,n_3,0} = \mu_{113} Q_{1,0,n_3-1,0}$$
$$+\mu_2 B_2 Q_{0,1,n_3,0} + \mu_2 B_{23} Q_{0,1,n_3-1,0}$$
$$+\mu_3 C_3 Q_{0,0,n_3+1,0} + \mu_4 D_4 Q_{0,0,n_3,1}$$
$$+\mu_4 D_{43} Q_{0,0,n_3-1,1} \quad \text{…(15)}$$

For $n_1 = n_2 = n_3 = n_4 = 0$

$\lambda Q_{0,\,0,\,0} = \mu_2\, B_2\, Q_{0,\,1,\,0\,0} + \mu_3 C_3\, Q_{0,\,0,\,1,\,0}$

$$+\mu_4 D_4 Q_{0,0,0,1} \quad \text{…(16)}$$

$$Q_{n_1 n_2 n_3 n_4} = \begin{cases} 1 & ; \quad n_1, n_2, n_3, n_4 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(X,Y,Z,R) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} Q_{n_1 n_2 n_3 n_4} X^{n_1} . Y^{n_2} . Z^{n_3} R^{n_4}$$

where $|X| = |Y| = |Z| = |R| = 1$ …(17)

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$$G_{n_2n_3n_4}(X)=\sum_{n_1=0}^{\infty} Q_{n_1n_2n_3n_4}X^{n_1}$$

$$F(X,Y,Z,R)=\sum Q_{n_1n_2n_3n_4}X^{n_1}Y^{n_2}Z^{n_3}R^{n_4}$$

we further define :

$$G_{n_2,n_3,n_4}(X)=\sum_{n_1=0}^{\infty} Q_{n_1n_2n_3n_4}X^{n_1} \quad \text{…(17A)}$$

$$G_{n_3,n_4}(X,Y)=\sum_{n_2=0}^{\infty} G_{n_2,n_3,n_4}(X)\, Y^{n_2} \quad \text{…(17B)}$$

$$G_{n_4}(X,Y,Z)=\sum_{n_3=0}^{\infty} G_{n_3,n_4}(X,Y)Z^{n_3} \quad \text{…(17C)}$$

$$F(X,Y,Z,R)=\sum_{n_4=0}^{\infty} G_{n_4}(X,Y,Z)R^{n_4} \quad \text{…(17D)}$$

Solving the above equations and using generating functions we have:

$$G_1=\frac{\begin{array}{l}\mu_1B_{21}(A_{12}(1-C_{34}B_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_1(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_1C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})B_{23}D_{42})+\lambda(C_{32}(-B_{23}D_{43}-B_{23})+B_{23}(-C_{34}-1)D_{42})\\ +\mu_1(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{14})B_{23}C_{32})D_{41}\end{array}}{\begin{array}{l}\mu_1B_{21}(A_{12}(1-C_{34}B_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_1(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_1C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})B_{23}D_{42})+\mu_1(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{4})B_{23}C_{32})D_{41}\end{array}}$$

…(18)

$$G_2=\frac{\begin{array}{l}\mu_2B_{21}(A_{12}(1-C_{34}D_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_3(A_{12}(1-C_{34}D_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42}\\ +\mu_2(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_2C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})B_{23}D_{42})+\mu_2(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{14})B_{23}C_{32})D_{41}\end{array}}{\begin{array}{l}\mu_2B_{21}(A_{12}(1-C_{34}D_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_2(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_2C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})(B_{23}D_{42})+\mu_2(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{14})B_{23}C_{32})D_{41}\end{array}}$$

…(19)

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$$G_3 = \frac{\begin{array}{l}\mu_3 B_{21}(A_{12}(1-C_{34}D_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_3(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_3 C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})B_{23}D_{42})+\lambda(A_{12}B_{23}D_{43}+B_{23})+(A_{14}-A_{13})B_{23}D_{42})\\ +\mu_3(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{14})B_{23}C_{32})D_{41}\end{array}}{\begin{array}{l}\mu_2 B_{21}(A_{12}(1-C_{34}D_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_3(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_3 C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})(B_{23}D_{42})+\mu_3(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{14})B_{23}C_{32})D_{41}\end{array}}$$

…(20)

$$G_4 = \frac{\begin{array}{l}\mu_4 B_{21}(A_{12}(1-C_{34}D_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_4(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_4 C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})B_{23}D_{42})+\mu_4(A_{12}B_{23}(C_{34}+1)+(A_{14}-A_{13})B_{23}C_{32})D_{41}\\ +\lambda(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{14})B_{23}C_{32})\end{array}}{\begin{array}{l}\mu_4 B_{21}(A_{12}(1-C_{34}D_{43})+C_{32}(A_{14}D_{43}+A_{13})+(A_{13}C_{34}+A_{14})D_{42})\\ +\mu_4(C_{32}(B_{23}D_{43}+B_{23})+B_{23}(C_{34}+1)D_{42})+\mu_4 C_{31}(A_{12}(B_{23}D_{43}+B_{23})\\ +(A_{14}-A_{13})(B_{23}D_{42})+\mu_4(A_{12}B_{23}(C_{34}+1)+(A_{13}-A_{14})B_{23}C_{32})D_{41}\end{array}}$$

…. (21)

let us denote;

$$\begin{aligned} f = {} & \mu_1 G_0(Y,Z,R)\left[1-\frac{1}{X}[A_{12}Y+A_{13}Z+A_{14}R\right] \\ & +\mu_2 G_0(X,Z,R)\left[1-\frac{1}{Y}(B_2+B_{21}X+B_{23}Z+B_{24}R)\right] \\ & +\mu_3 G_0(X,Y,R)\left[1-\frac{1}{Z}(C_3+C_{31}X+C_{32}Y+C_{34}R)\right] \\ & +\mu_4 G_0(X,Y,Z)\left[1-\frac{1}{R}(D_4+D_{41}X+D_{42}Y+D_{43}Z)\right] \end{aligned}$$

and

$$\begin{aligned} g = {} & \lambda(1-X)+\mu_1\left[1-\frac{1}{X}(A_{12}Y+A_{13}Z+A_{14}R)\right] \\ & +\mu_2\left[1-\frac{1}{Y}(B_2+B_{21}X+B_{23}Z+B_{24}R)\right] \\ & +\mu_3\left[1-\frac{1}{Z}(C_3+C_{31}X+C_{32}Y+C_{34}R)\right] \\ & +\mu_4\left[1-\frac{1}{R}(D_4+D_{41}X+D_{42}Y+D_{43}Z)\right] \end{aligned}$$

Let Lq_1 denote the mean queue length at the 1st server S_1.

$$Lq_1 = \frac{\left(\frac{\partial f}{\partial X}\right)_{(1,1,1,1)}\left(\frac{\partial^2 g}{\partial X^2}\right)_{(1,1,1,1)}+\left(\frac{\partial g}{\partial X}\right)_{(1,1,1,1)}\left(\frac{\partial^2 f}{\partial X^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial X}\right)_{(1,1,1,1)}\right]^2}$$

$$Lq_1 = \frac{\begin{array}{c}(\mu_1 G_1 - \mu_2 B_{21} G_2 - \mu_3 C_{31} G_3 - \mu_4 D_{41} G_4)(-2\mu_1) \\ +(\lambda + \mu_1 - B_{21}\mu_2 - C_{31}\mu_3 - D_{41}\mu_4)(-2\mu_1 G_1)\end{array}}{1[-\lambda + \mu_1 - B_{21}\mu_2 - C_{31}\mu_3 - D_{41}\mu_4]^2}$$

or $Lq_1 = -\mu_1 \left[\frac{(\mu_1 G_1 - \mu_2 B_{21} G_2 - \mu_3 C_{31} G_3 - \mu_4 D_{41} G_4)}{(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})^2} + \frac{G_1}{(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})}\right]$

…(22)

Let Lq_2 denote the mean queue length at the 2nd server S_2.

$$Lq_2 = \frac{\left(\frac{\partial f}{\partial Y}\right)_{(1,1,1,1)} \left(\frac{\partial^2 g}{\partial Y^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial Y}\right)_{(1,1,1,1)} \left(\frac{\partial^2 f}{\partial Y^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial Y}\right)_{(1,1,1,1)}\right]^2}$$

$$Lq_2 = \frac{\begin{array}{c}-(\mu_1 A_{12} G_1 + \mu_2 G_2 - \mu_3 G_3 C_{32} - \mu_4 G_4 D_{42})(-2\mu_2) + \\ (-\mu_1 A_{12} + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})(-2\mu_2 G_2)\end{array}}{2[(-\mu_1 A_{12} + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})]^2}$$

$Lq_2 = -\mu_2 \left[\frac{(-\mu_1 A_2 G_1 + \mu_2 G_2 - \mu_3 G_3 C_{32} - \mu_4 G_4 D_{42})}{(-\mu_1 A_{12} + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})^2} + \frac{G_2}{(-\mu_1 A_{12} + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})}\right]$

…(23)

Let Lq_3 denote the mean queue length at the 3rd server S_3.

$$Lq_3 = \frac{\left(\frac{\partial f}{\partial Z}\right)_{(1,1,1,1)} \left(\frac{\partial^2 g}{\partial Z^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial Z}\right)_{(1,1,1,1)} \left(\frac{\partial^2 f}{\partial Z^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial Z}\right)_{(1,1,1,1)}\right]^2}$$

$Lq_3 = \frac{\begin{array}{c}(-\mu_1 G_1 A_{13} - \mu_2 G_2 B_{23} + \mu_3 G_3 - \mu_4 G_4 D_{43})(-2\mu_3) + \\ (-\mu_1 A_{13} - \mu_2 B_{23} + \mu_3 - \mu_4 D_{43})(-2\mu_3 G_3)\end{array}}{1[(-\mu_1 A_3 - \mu_2 B_{23} + \mu_3 - \mu_4 D_{43})]^2}$

$Lq_3 = -\mu_3 \left[\frac{(-\mu_1 G_1 A_{13} - \mu_2 G_2 B_{23} + \mu_3 G_3 - \mu_4 G_4 D_{43})}{(-\mu_1 A_{13} - \mu_2 B_{23} + \mu_3 - \mu_4 D_{43})^2} + \frac{G_3}{(-\mu_1 A_{13} - \mu_2 B_{23} + \mu_3 - \mu_4 D_{43})}\right]$

…(24)

Let Lq_4 denote the mean queue length at the 4th server S_4.

$$Lq_4 = \frac{\left(\frac{\partial f}{\partial R}\right)_{(1,1,1,1)} \left(\frac{\partial^2 g}{\partial R^2}\right)_{(1,1,1,1)} + \left(\frac{\partial g}{\partial R}\right)_{(1,1,1,1)} \left(\frac{\partial^2 f}{\partial R^2}\right)_{(1,1,1,1)}}{2\left[\left(\frac{\partial g}{\partial R}\right)_{(1,1,1,1)}\right]^2}$$

$$Lq_4 = \frac{\begin{array}{c}(-\mu_1 G_1 A_{14} - \mu_2 G_2 B_{24} - \mu_3 G_3 C_{34} + \mu_4 G_4)(-2\mu_4) \\ +(-\mu_1 A_{14} - \mu_2 B_{24} - \mu_3 C_{34} + \mu_4)(-2\mu_4 G_4)\end{array}}{2[(-\mu_1 A_{14} - \mu_2 B_{24} - \mu_3 C_{34} + \mu_4)]^2}$$

$Lq_4 = -\mu_4 \left[\frac{(-\mu_1 G_1 A_{14} - \mu_2 G_2 B_{24} - \mu_3 G_3 C_{34} + \mu_4 G_4)}{(-\mu_1 A_{14} - \mu_2 B_{24} - \mu_3 C_{34} + \mu_4)^2} + \frac{G_4}{(-\mu_1 A_{14} - \mu_2 B_{24} - \mu_3 C_{34} + \mu_4)}\right]$

… (25)

If Lq be the mean queue length of the whole system then

$Lq = Lq_1 + Lq_2 + Lq_3 + Lq_4$

From (22), (23), (24) and (25) we have

$$\begin{aligned} Lq = -\mu_1 &\left[\frac{(\mu_1 G_1 - \mu_2 B_{21} G_2 - \mu_3 C_{31} G_3 - \mu_4 D_{41} G_4)}{(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})^2} + \frac{G_1}{(-\lambda + \mu_1 - \mu_2 B_{21} - \mu_3 C_{31} - \mu_4 D_{41})}\right] \\ &+ -\mu_2 \left[\frac{(-\mu_1 A_2 G_1 + \mu_2 G_2 - \mu_3 G_3 C_{32} - \mu_4 G_4 D_{42})}{(-\mu_1 A_{12} + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})^2} + \frac{G_2}{(-\mu_1 A_{12} + \mu_2 - \mu_3 C_{32} - \mu_4 D_{42})}\right] \\ &- \mu_3 \left[\frac{(-\mu_1 G_1 A_{13} - \mu_2 G_2 B_{23} + \mu_3 G_3 - \mu_4 G_4 D_{43})}{(-\mu_1 A_{13} - \mu_2 B_{23} + \mu_3 - \mu_4 D_{43})^2} + \frac{G_3}{(-\mu_1 A_{13} - \mu_2 B_{23} + \mu_3 - \mu_4 D_{43})}\right] \\ &+ -\mu_4 \left[\frac{(-\mu_1 G_1 A_{14} - \mu_2 G_2 B_{24} - \mu_3 G_3 C_{34} + \mu_4 G_4)}{(-\mu_1 A_{14} - \mu_2 B_{24} - \mu_3 C_{34} + \mu_4)^2} + \frac{G_4}{(-\mu_1 A_{14} - \mu_2 B_{24} - \mu_3 C_{34} + \mu_4)}\right] \end{aligned}$$

… (26)

where G_1, G_2, G_3 and G_4 are given by (18), (19), (20) and (21).

4. Conclusion

Thus, we can compute the mean queue length of the feedback queuing system with four servers and a limited chance of customer's revisits to any server using (26). A queue management system helps businesses by increasing productivity, doing away with the logistical needs of a physical queue (such as floor space, barrier costs, and customer traffic flow etc.) and providing useful information based on user data.

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