

Inverse Domination Numbers to the Kronecker Product of Some Standard graphs with Path

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ABSTRACT

Let G be a non-trivial connected graph with vertex set $V(G)$. A subset D of $V(G)$ is said to be a dominating set if all the elements of $V-D$ are adjacent with at least one element in D . A minimum dominating set is a dominating set with minimum cardinality and its value is known as its domination number, denoted by $\gamma(G)$.

If D' is a dominating set in $V-D$ is called the inverse dominating set of G with respect to D . The minimum cardinality taken over all inverse dominating sets of G is called the inverse domination number denoted by γ' . In this paper we find the inverse domination number of the resultant graphs obtained by the Kronecker product of some standard graphs with path.

Keywords: Domination, inverse domination, Kronecker product.

1. Introduction

The theory of domination began around 1960. In 1958, Berge formalized this concept. In 1962 O. Ore has used the term dominating set and domination number for the same concept in Graph Theory. This theory has been used in document summarization and designing secure for electrical grids. In 1991, the research paper of Kulli and Sigarkanti help to study the concept of inverse domination.

The Kronecker product $G_1(K)G_2$ of two graphs G_1 and G_2 is defined as $V(G_1(K)G_2) = V(G_1) \times V(G_2)$ and $E(G_1(K)G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. It is also named as the direct product and Tensor product.

Definition: 1.1

A walk in G is a finite non-null sequence $W = v_0e_1v_1e_2v_2, \dots, e_kv_kv_k$ whose terms are alternately vertices and edges. A walk is called a path if the vertices are distinct. A closed path is called a cycle. Number of edges in a cycle is called its length. Path and cycle of n vertices respectively denoted by P_n and C_n .

Definition: 1: 2

The open neighborhood of $v \in V(G)$ is the set of all vertices which are adjacent to v . That is $N(v) = \{u/uv \in E(G) \text{ and } u \in V(G)\}$, closed neighborhood of $v \in V(G)$ is defined as $N[v] = N(v) \cup \{v\}$.

Definition: 1:3

The upper and lower sealing of x is defined as

$$[x] = \begin{cases} x & \text{if } x \in N \\ [x+1] & \text{otherwise} \end{cases} ; \quad [x] = \begin{cases} x & \text{if } x \in N \\ [x] & \text{otherwise} \end{cases}$$

Preliminaries

Result 1:

If $G = P_n$ then $\gamma'(G) = \lceil \frac{n}{3} \rceil$ for all $n > 3$

Result 2:

If $G = C_n$ then $\gamma'(G) = \gamma(G) = k$ if $n \equiv 0 \pmod{3}$ & $n = 3k$

Theorem 1.1

Let G_1 and G_2 be any two paths of length m and n respectively. $G = G_1 (K) G_2$ then the domination and inverse domination number of G is

$$\gamma(G) = \begin{cases} n \lceil \frac{m}{3} \rceil & \text{if } m \equiv 0 \pmod{3} \\ n \lceil \frac{m}{3} + 1 \rceil & \text{if } m \equiv 1 \pmod{3} \\ n \lceil \frac{m}{3} \rceil & \text{if } m \equiv 2 \pmod{3} \end{cases}$$

$$\gamma'(G) = \begin{cases} n \lceil \frac{m}{3} + 1 \rceil & \text{if } m \equiv 0 \pmod{3} \\ n \lceil \frac{m}{3} \rceil & \text{if } m \equiv 1, 2 \pmod{3} \text{ for all } m, n \text{ and } m \geq n. \end{cases}$$

Proof:

Let G_1 and G_2 be the path of length m and n respectively such that $m \geq n$.

Let $V(G_1) = \{u_i / 1 \leq i \leq m\}$; $V(G_2) = \{v_j / 1 \leq j \leq n\}$ and $d(u_1) = d(u_m) = d(v_1) = d(v_n) = 1$

G is the Kronecker product of G_1 and G_2 .

Now $V(G) = \{u_i v_j / 1 \leq i \leq m; 1 \leq j \leq n\}$ with

$$d(u_1 v_1) = d(u_1 v_n) = d(u_m v_1) = d(u_m v_n) = 1$$

$$d(u_i v_j) = d(u_i v_1) = d(u_i v_n) = d(u_m v_j) = 2 \text{ for all } 1 < i < m; 1 < j < n$$

$$d(u_i v_j) = 4 \text{ for all } 2 \leq i < m; 2 \leq j < n$$

Case (i):

If $m \equiv 0 \pmod{3}$

Choose D and D' are the subsets of $V(G)$ such that

$$D = \left\{ u_{3i-1} v_j / 1 \leq j \leq n; 1 \leq i \leq \lceil \frac{m}{3} \rceil \right\} \text{ and}$$

$$D' = \left\{ u_{3i-2} v_j; u_n v_j / 1 \leq j \leq n; 1 \leq i \leq \lceil \frac{m}{3} \rceil \right\}$$

Clearly $N[D] = N[D'] = V(G)$ is the required minimum dominating and inverse dominating sets of $V(G)$.

$$\Rightarrow |D| = n \left\lceil \frac{m}{3} \right\rceil \& |D'| = (n+1) \left\lceil \frac{m}{3} \right\rceil \quad \dots\dots (i)$$

If $m \equiv 1 \pmod{3}$

Choose D and D' such that

$$D = \left\{ u_{3i-1} v_j ; u_n v_j / 1 \leq j \leq n ; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$$

$$D' = \left\{ u_{3i-2} v_j / 1 \leq j \leq n ; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$$

where D and D' are the required dominating and inverse dominating sets with

$$|D| = n \left\lceil \frac{m}{3} \right\rceil \& |D'| = n \left\lceil \frac{m}{3} \right\rceil \quad \dots\dots (ii)$$

If $m \equiv 2 \pmod{3}$

Choose the subsets D and D' of $V(G)$ such that

$$D = \left\{ u_{3i-1} v_j / 1 \leq j \leq n ; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$$

$$D' = \left\{ u_{3i-2} v_j / 1 \leq j \leq n ; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\}$$

Now D and D' are the minimum sets of $V(G)$ such that

$$N[D] = N[D'] = V(G) \text{ and}$$

$$|D| = n \left\lceil \frac{m}{3} \right\rceil \text{ and } |D'| = n \left\lceil \frac{m}{3} \right\rceil \quad \dots\dots (iii)$$

From the above equations we get

$$\gamma(G) = \begin{cases} n \left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 0 \pmod{3} \\ n \left\lceil \frac{m}{3} \right\rceil + 1 & \text{if } m \equiv 1 \pmod{3} \\ n \left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 2 \pmod{3} \end{cases}$$

$$\gamma'(G) = \begin{cases} n \left\lceil \frac{m}{3} \right\rceil + 1 & \text{if } m \equiv 0 \pmod{3} \\ n \left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 1, 2 \pmod{3} \end{cases}$$

Hence the proof

Result 1.2

If $G_1 = P_m, G_2 = P_n, G = G_1(K) G_2$ then the domination and inverse domination numbers of G is $\gamma(G) \leq n \left\lceil \frac{m}{3} \right\rceil$ and $\gamma'(G) = n \left\lceil \frac{m}{3} \right\rceil + 1$ for all $m > n$

Theorem 1.3

Let G be the Kronecker product of cycle of order m and a path of order n then the domination and inverse domination numbers is given as

$$\gamma(G) = \gamma'(G) = \begin{cases} n \left\lceil \frac{m}{3} \right\rceil & \text{if } m \equiv 0 \pmod{3} \\ n \left\lceil \frac{m}{3} \right\rceil + 1 & \text{otherwise} \end{cases}$$

Proof :

Let G be the Kronecker product of C_m and P_n . The vertex sets of C_m and P_n are taken as

$V(C_m) = \{ u_i / 1 \leq i \leq m \}$ and $V(P_n) = \{ v_j / 1 \leq j \leq n \}$ with

$d(u_i) = 2$ for all i , u_i adjacent with u_{i-1} and u_{i+1} ; $1 < i \leq m-1$

$d(v_1) = d(v_n) = 1$ and $N(v_i) = \{ v_{i-1}, v_{i+1} \} / 1 \leq i \leq n$; i modulus n

Let $V(G) = \{ u_i v_j / 1 \leq i \leq m, 1 \leq j \leq n \}$ with

$d(u_i v_1) = d(u_i v_n) = 2$ for all $1 \leq i \leq m$ and

$d(u_i v_j) = 4$ for all $1 < i < m$ and $1 < j < n$

Case (i) If $n \equiv 0 \pmod{3}$

$$\begin{aligned} \text{Choose } D &= \left\{ u_{3i-2} v_j / 1 \leq j \leq n; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\} \\ D' &= \left\{ u_{3i-1} v_j / 1 \leq j \leq n; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\} \end{aligned} \quad \dots\dots (i)$$

Case (ii) If $n \equiv 1, 2 \pmod{3}$

$$\begin{aligned} \text{Choose } D &= \left\{ u_{3i-2} v_j; u_{n-1} v_j / 1 \leq j \leq n; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\} \\ D' &= \left\{ u_{3i-1} v_j; u_n v_j / 1 \leq j \leq n; 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \right\} \end{aligned} \quad \dots\dots (ii)$$

In both cases the minimum disjoint sets D and D' satisfy

$$N[D] = N[D'] = V(G).$$

$\Rightarrow D$ and D' are the required minimum dominating and inverse dominating set of $V(G)$ with

$$|D| = |D'| = n \left\lceil \frac{m}{3} \right\rceil \text{ if } m \equiv 0 \pmod{3} \text{ and}$$

$$|D| = |D'| = n \left\lceil \frac{m}{3} \right\rceil + 1 \text{ if } m \equiv 1, 2 \pmod{3}.$$

$$\Rightarrow \gamma(G) = \gamma'(G) = \begin{cases} n \left\lfloor \frac{m}{3} \right\rfloor & m \equiv 0 \pmod{3} \\ n \left\lceil \frac{m}{3} \right\rceil & \text{otherwise} \end{cases}$$

Theorem 1.4

Let G_1 and G_2 are the cycles of order m and n respectively. G be the Kronecker product of G_1 and G_2 then $\gamma(G) = \gamma'(G) = \frac{mn}{4}$ where $m, n \equiv 0 \pmod{4}$

Proof :

Let $V(G_1) = \{ u_i / 1 \leq i \leq m \}$ and $V(G_2) = \{ v_j / 1 \leq j \leq n \}$ are the vertex sets of G_1 and G_2 respectively. $G = G_1(K)G_2$.

Now, $V(G) = \{ u_i v_j / 1 \leq i \leq m \text{ and } 1 \leq j \leq n \}$.

Clearly, $d(u_i v_j) = 4$ and $N(u_i v_j) = \{ u_{i-1} v_{j-1}, u_{i-1} v_{j+1}, u_{i+1} v_{j-1}, u_{i+1} v_{j+1} \}$ for all i, j

Choose D and D' are the subset of $V(G)$ such that

$$D = \left\{ u_{4i-2} v_{4j-3}; u_{4i-2} v_{4j-2}; u_{4i-1} v_{4j-3}; u_{4i-1} v_{4j-2} / 1 \leq i \leq \frac{n}{4} \text{ for each } i, 1 \leq i \leq \frac{m}{4} \right\}$$

$$D' = \left\{ u_{4i-2} v_{4j-1}; u_{4i-2} v_{4j}; u_{4i-1} v_{4j-1}; u_{4i-1} v_{4j} / 1 \leq j \leq \frac{n}{4} \text{ for each } i, 1 \leq i \leq \frac{m}{4} \right\}$$

Clearly, D and D' are disjoint minimum subsets of $V(G)$ satisfies

$$N[D] = N[D'] = V(G).$$

Hence D and D' are the required dominating and inverse dominating sets of $V(G)$ with

$$|D| = |D'| = \frac{mn}{4}$$

$$\Rightarrow \gamma(G) = \gamma'(G) = \frac{mn}{4}$$

Theorem 1.5

G_1 is a path graph of order m and G_2 be a complete graph of order n , then G is the Kronecker product of G_1 and G_2 then $\gamma(G) = \gamma'(G) = m$.

Proof :

Let $V(G_1) = \{ u_i / 1 \leq i \leq m \}$ and $V(G_2) = \{ v_j / 1 \leq j \leq n \}$ be the vertex sets of P_m and K_n respectively.

$V(G) = \{ u_i v_j / 1 \leq i \leq m ; 1 \leq j \leq n \}$ be the vertex set of G ; where $G = P_m(K)K_n$ with

$d(u_i v_j) = 2(n - i)$ for all $i; 1 < i < m$ and $d(u_i v_j) = n - 1$ for each $i = 1, m$.

Clearly, $N(u_i v_j) = \{ u_{i-1} v_k, u_{i+1} v_k / j \neq k \}$.

Let $D_{ik} = \{ u_i v_k / 1 \leq i \leq m \} : 1 \leq k \leq n$ (i)

$D_{kj} = \{ u_k v_j / 1 \leq j \leq n \} , 1 \leq k \leq m.$ (ii)

From eqn (i)-for any fixed k , no two elements of D_{ik} are adjacent in G . In similar, from eqn (ii) for any fixed k , no two elements of D_{kj} are adjacent in G .

Choose $D_{ik} \subseteq V(G)$ such that

$$D_{i1} = \{ u_i v_1 / 1 \leq i \leq m \}$$

$$D_{i2} = \{ u_i v_2 / 1 \leq i \leq m \}$$

$$D_{in} = \{ u_i v_n / 1 \leq i \leq m \} \text{ satisfies}$$

$$N[D_{i1}] = N[D_{i2}] = \dots = N[D_{in}] = V(G).$$

Hence, each $D_{ik}, 1 \leq k \leq n$, is the minimum set which satisfies the condition

$$N[D_{ik}] = V(G) \text{ for all } i = 1, 2, \dots, m \text{ for fixed } k = 1, 2, \dots, n.$$

Also $D_{ik} \cap D_{it} = \emptyset$ if $k \neq t$

Hence, every $D_{ik} \subseteq V(G)$ for any fixed $k = 1, 2, \dots, n$ is the required dominating set and another $D_{it} \subseteq V(G)$ such that $1 \leq t \leq n$ and $t \neq k$ are the dominating and inverse dominating sets of $V(G)$.

Therefore, each dominating set D_{ik} for any fixed k of $V(G)$ we have $(n-1)$ inverse dominating sets with same cardinality.

That is, $D_{i1}, 1 \leq i \leq m$, is a dominating set of $V(G)$,

then $D_{ik}, 1 \leq i \leq m, 1 < k \leq n$, are the inverse dominating sets of G .

$$\text{Hence, } \gamma(G) = \gamma'(G) = m.$$

Hence, the result.

Theorem 1.6

Graph G is the Kronecker product of C_m and $K_{1,n}$ then the domination and inverse domination numbers is given by

$$\gamma(G) = \begin{cases} m, & \text{if } m \text{ is even and } m \equiv 0 \pmod{4} \\ 2 \left\{ \left\lceil \frac{m}{4} \right\rceil + \left\lceil \frac{m}{4} \right\rceil \right\}, & \text{if } m \text{ is even and } \frac{m}{2} \text{ is odd} \\ \left\lceil \frac{m}{4} \right\rceil + 3 \left\lceil \frac{m}{4} \right\rceil, & \text{if } m \equiv 1 \pmod{4} \\ 3 \left\lceil \frac{m}{4} \right\rceil + \left\lceil \frac{m}{4} \right\rceil, & \text{if } m \equiv 3 \pmod{4} \end{cases}$$

$$\gamma'(G) = \begin{cases} m, & \text{if } m \text{ is even and } m \equiv 0 \pmod{4} \\ 4 \left\lfloor \frac{m}{4} \right\rfloor + 2(n-1), & \text{if } m \text{ is even and } \frac{m}{2} \text{ is odd} \\ \left\lfloor \frac{m}{4} \right\rfloor + n - 1 & \text{if } m \equiv 1 \pmod{4} \\ \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor + n & \text{if } m \equiv 3 \pmod{4} \end{cases}$$

Proof:

Let C_m and $K_{1,n}$ are the cycle of order m and complete bipartite graph of order $n+1$ respectively. with

$$V(C_m) = \{u_i / 1 \leq i \leq m\}, V(K_{1,n}) = \{v_j / 1 \leq j \leq n\} \text{ Such that } \deg(v_0) = n.$$

Let G be the Kronecker product of C_m and $K_{1,n}$ then

$$V(G) = \{u_i v_j / 1 \leq i \leq m, 0 \leq j \leq n\} \text{ such that}$$

Proof:

Let C_m and $K_{1,n}$ are the cycle of order m and complete bipartite graph of order $n+1$ respectively. with

$$V(C_m) = \{u_i / 1 \leq i \leq m\}, V(K_{1,n}) = \{v_j / 1 \leq j \leq n\} \text{ Such that } \deg(v_0) = n.$$

Let G be the Kronecker product of C_m and $K_{1,n}$ then

$$V(G) = \{u_i v_j / 1 \leq i \leq m, 0 \leq j \leq n\} \text{ such that}$$

$$d(u_i v_0) = 2n \text{ and } d(u_i v_j) = 2 \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n.$$

Case (i): m is even and $m \equiv 0 \pmod{4}$

then G is two disjoint components G_1 and G_2 with

$$V(G_1) = \{u_{2i} v_0; u_{2i-1} v_j / 1 \leq i \leq \frac{m}{2}; 1 \leq j \leq n\} \text{ and}$$

$$V(G_2) = \{u_{2i-1} v_0; u_{2i} v_j / 1 \leq i \leq \frac{m}{2}; 1 \leq j \leq n\} \text{ with}$$

$$d(u_{2i} v_0) = d(u_{2i-1} v_0) = 2n \text{ and } d(u_{2i-1} v_j) = d(u_{2i} v_j) = 2 \text{ for all } 1 \leq i \leq \frac{m}{2}; 1 \leq j \leq n.$$

Since G_1 and G_2 are the disjoint components of G .

Hence, dominating and inverse dominating sets of G_1 and G_2 respectively are the dominating and inverse dominating sets of G .

Choose D_1 and D_1' are the disjoint minimum subsets of $V(G_1)$ such that

$$\left. \begin{aligned} D_1 &= \left\{ u_{4i-2} v_0; u_{4i-3} v_1 / 1 \leq i \leq \frac{m}{4} \right\} \\ D_1' &= \left\{ u_{4i-2} v_0; u_{4i-3} v_1 / 1 \leq i \leq \frac{m}{4} \right\} \end{aligned} \right\} \dots\dots(i)$$

are the minimum subsets of G_1 which satisfy $N[D_1] = N[D_1'] = V(G_1)$

In similar choose D_2 and D_2' such that

$$\left. \begin{aligned} D_2 &= \left\{ u_{4i-3} v_0 ; u_{4i-2} v_1 / 1 \leq i \leq \frac{m}{4} \right\} \& \\ D_2' &= \left\{ u_{4i-1} v_0 ; u_{4i} v_1 / 1 \leq i \leq \frac{m}{4} \right\} \end{aligned} \right\} \dots\dots (ii)$$

are the minimum disjoint subsets of G_2 satisfies

$$N[D_2] = N[D_2'] = V(G_2).$$

Therefore, $D = D_1 \cup D_2$

$$D = \left\{ u_{4i-2} v_0 ; u_{4i-3} v_0 ; u_{4i-3} v_1 ; u_{4i-2} v_1 / 1 \leq i \leq \frac{m}{4} \right\} \text{ and}$$

$$D' = D_1' \cup D_2'$$

$$D' = \left\{ u_{4i} v_0 ; u_{4i-1} v_1 ; u_{4i-1} v_0 ; u_{4i} v_1 / 1 \leq i \leq \frac{m}{4} \right\}$$

are the required minimum dominating and inverse dominating sets of G with

$$|D| = |D'| = 4 \left(\frac{m}{4} \right)$$

$$\Rightarrow |D| = |D'| = m \dots\dots (a)$$

Case (ii): If m is even and $\frac{m}{2}$ is odd.

In this case, we get the graph G as the two disjoint components of G_1 and G_2 , each components having odd number of vertices with degree $2n$.

$$\text{Let } V(G_1) = \{ u_{2i} v_0 ; u_{2i-1} v_j / 1 \leq i \leq \frac{m}{2} ; 1 \leq j \leq n \} \text{ and}$$

$$V(G_2) = \{ u_{2i-1} v_0 u_{2i} v_j / 1 \leq i \leq \frac{m}{2} ; 1 \leq j \leq n \}$$

Since G is the union of G_1 and G_2 , clearly dominating and inverse dominating sets of G is the union of the dominating and inverse dominating sets of G_1 and G_2 respectively.

Choose D_1 and D_1' are the disjoint subsets of $V(G_1)$ such that

$$\left. \begin{aligned} D_1 &= \left\{ u_{4i-2} v_0 ; u_{4k+1} v_1 / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor ; 1 \leq k \leq \left\lfloor \frac{m}{4} \right\rfloor \right\} \& \\ D_1' &= \left\{ u_{4i} v_0 ; u_{4k-1} v_1 ; u_{4i} v_j / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor ; 1 \leq k \leq \left\lfloor \frac{m}{4} \right\rfloor - 1 ; 1 \leq j \leq n \right\} \end{aligned} \right\} \dots\dots (iii)$$

Clearly, D_1 and D_1' are minimum sets satisfies

$$N[D_1] = N[D_1'] = V(G_1).$$

Hence, D_1 and D_1' are the minimum dominating sets of G_1 .

In similar, choose D_2 and D_2' are the minimum disjoint subsets of G_2 .

$$D_2 = \left\{ u_{4i-3} v_0 ; u_{4k} v_1 / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor ; 1 \leq k \leq \left\lfloor \frac{m}{4} \right\rfloor \right\} \& \dots\dots (iv)$$

$$D_2' = \{ u_{4i-1}v_0, u_{4k+2}v_1; u_nv_j / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor; 1 \leq k \leq \left\lfloor \frac{m}{4} \right\rfloor - 1; 1 \leq j \leq n \}$$

Clearly the sets D_2 and D_2' satisfies

$$N[D_2] = N[D_2'] = V(G_2)$$

Hence, D_2 and D_2' are the minimum dominating and inverse dominating sets of G_2 .

Hence, $D = D_1 \cup D_2$

$$D = \left\{ u_{4i-2}v_0; u_{4k+1}v_1; u_{4i-3}v_0; u_{4k}v_1 / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor; 1 \leq k \leq \left\lfloor \frac{m}{4} \right\rfloor \right\} \text{ and } \dots(v)$$

$$D_2' = \left\{ u_{4i}v_0; u_{4k-1}v_1; u_{4i-1}v_0; u_{4k+2}v_1; u_nv_j; 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor; 1 \leq k \leq \left\lfloor \frac{m}{4} \right\rfloor - 1; 1 \leq j \leq n \right\}$$

are the required dominating and inverse dominating sets of G .

$$\left. \begin{aligned} |D| &= 2 \left\{ \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor \right\} \text{ and} \\ |D'| &= 2 \left\lfloor \frac{m}{4} \right\rfloor + 2 \left\{ \left\lfloor \frac{m}{4} \right\rfloor - 1 \right\} + 2n \\ &= 4 \left\lfloor \frac{m}{4} \right\rfloor + 2(n-1) \end{aligned} \right\} \dots\dots \textcircled{b}$$

Case (iii) If m is odd, and $m \equiv 1 \pmod{4}$

for all odd m , we get G is a connected simple graph with

$$V(G) = \{ u_iv_j / 1 \leq i \leq m; 0 \leq j \leq n \} \text{ with}$$

$$d(u_iv_0) = 2n; 1 \leq i \leq m; d(u_iv_j) = 2 \text{ for all } 1 \leq i \leq m; 1 \leq j \leq n$$

$$\text{also, } N(u_{2i}v_0) = \{ u_{2i-1}v_j; u_{2i+1}v_j / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor; 1 \leq j \leq n \} \text{ and}$$

$$N(u_{2i-1}v_0) = \{ u_{2i-2}v_j; u_{2i}v_j / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor; 1 \leq j \leq n \}$$

$$\text{Also, } N(u_1v_0) = \{ u_mv_j; u_2v_j / 1 \leq j \leq n \}$$

In this case we choose D and D' as

$$D = \left\{ u_{4i-3}v_0; u_{4j}v_0; u_{4k-2}v_1; u_{4l-3}v_1 / 1 \leq i \leq \left\lfloor \frac{m}{4} \right\rfloor; 1 \leq j; k; l \leq \left\lfloor \frac{m}{4} \right\rfloor - \right\} \dots\dots (vi)$$

$$D' = \left\{ u_{4i-1}v_0; u_{4i-2}v_0; u_{4j}v_1; u_{4k-1}v_1; u_nv_i / 1 \leq i; j \leq \left\lfloor \frac{m}{4} \right\rfloor; 1 \leq k \leq \left\lfloor \frac{m}{4} \right\rfloor - 1; 1 \leq i \leq n \right\}$$

Clearly D and D' are the disjoint subsets of $V(G)$ satisfies $N[D] = N[D'] = V(G)$ with

$$\left. \begin{aligned} |D| &= \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor - & \\ |D'| &= 3 \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor - 1 + n; & \\ |D'| &= 4 \left\lfloor \frac{m}{4} \right\rfloor + n - 1. & \end{aligned} \right\} \dots\dots (c)$$

(Case (iv) If m is odd and $m \equiv 3 \pmod{4}$)

In this case G is a connected graph similar to in Case (iii) but the dominating and inverse dominating sets are different from Case (iii).

Let $V(G) = \{ u_i v_j / 1 \leq i \leq m ; 0 \leq j \leq n \}$ with

Choose $D = \{ u_{4i-3} v_0, u_{4i-2} v_0, u_{4i-2} v_1, u_{4j-1} v_t / 1 \leq i \leq \lfloor \frac{m}{4} \rfloor ; 1 \leq j \leq \lfloor \frac{m}{4} \rfloor \}$ (vii)

and $D' = \{ u_{4i-1} v_0, u_{4j} v_0, u_{4i-3} v_1, u_{4i-3} v_1, u_m v_t / 1 \leq i \leq \lfloor \frac{m}{4} \rfloor ; 1 \leq j, l \leq \lfloor \frac{m}{4} \rfloor ; 1 \leq t \leq n \}$

Where D and D' are the minimum disjoint subsets of $V(G)$ with such that

$N[D] =$

$N[D'] = V(G)$ with

$$\left. \begin{aligned} |D| &= 3 \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor \\ |D'| &= \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor + n \end{aligned} \right\} \text{..... (d)}$$

From (a), (b), (c) and (d) we get

$$\gamma(G) = \begin{cases} m, & \text{if } m \text{ is even and } m \equiv 0 \pmod{4} \\ 2 \left\{ \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor \right\} & \text{if } m \text{ is even and } \frac{m}{2} \text{ is odd} \\ \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor & \text{if } m \equiv 1 \pmod{4} \\ \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor & \text{if } m \equiv 3 \pmod{4} \end{cases}$$

$$\gamma'(G) = \begin{cases} m & \text{if } m \text{ is even \& } m \equiv 0 \pmod{4} \\ 4 \left\lfloor \frac{m}{4} \right\rfloor 2(n-1) & \text{if } m \text{ is even and } \frac{m}{2} \text{ is odd} \\ \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor & \text{if } m \equiv 1 \pmod{4} \\ \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor & \text{if } m \equiv 3 \pmod{4} \end{cases}$$

Corollary 1.7

By Theorem 1.6, $GS = C_m(K) K_{1,n}$ then $\gamma(G) = m$ for all $m \geq 4$.

Proof :

By theorem 1.6

$$G = \begin{cases} m, & \text{if } m \text{ is even and } m \equiv 0 \pmod{4} \\ 2 \left\{ \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor \right\}, & \text{if } m \text{ is even and } \frac{m}{2} \text{ is odd} \\ \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor, & \text{if } m \equiv 1 \pmod{4} \\ 3 \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor, & \text{if } m \equiv 3 \pmod{4} \end{cases}$$

Case (i): If $m \equiv 0 \pmod{4}$, no need to prove.

Case (ii): If m is even and $\frac{m}{2}$ is odd

Let $m = 4k + 2$.

$$\gamma(G) = 2 \left\{ \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor \right\}$$

$$\begin{aligned}
 &= 2 \left\{ \left\lfloor \frac{4k+2}{4} \right\rfloor + \left\lfloor \frac{4k+2}{4} \right\rfloor \right\} & \{ \because \left\lfloor \frac{4k+2}{4} \right\rfloor + \frac{4k+4}{4} \& \left\lfloor \frac{4k+2}{4} \right\rfloor = \frac{4k}{4} \} \\
 &= 2 \left\{ \left\lfloor \frac{4k+2+2}{4} \right\rfloor + \left\lfloor \frac{4k}{4} \right\rfloor \right\} \\
 &= 2 \left\{ \left\lfloor \frac{8k+4}{4} \right\rfloor \right\} \\
 &= 2 \left\{ \left\lfloor \frac{2(4k+2)}{4} \right\rfloor \right\} \\
 &= 4k + 2 \\
 &= m. & \dots\dots (ii)
 \end{aligned}$$

Case (iii):

If $m \equiv 1 \pmod{4}$; let $m = 4k + 1$.

$$\begin{aligned}
 \gamma(G) &= \left\lfloor \frac{m}{4} \right\rfloor + 3 \left\lfloor \frac{m}{4} \right\rfloor \\
 &= \left\lfloor \frac{4k+1}{4} \right\rfloor + 3 \left\lfloor \frac{4k+1}{4} \right\rfloor & \{ \because [m \equiv 1 \pmod{4}] \\
 &= \frac{4k+1+3}{4} + 3 \left\lfloor \frac{4k}{4} \right\rfloor & 3 \left\lfloor \frac{4k+1}{4} \right\rfloor = \left\lfloor \frac{4k+1+3}{4} \right\rfloor = \left\lfloor \frac{4k+1}{4} \right\rfloor = \frac{4k}{4} \} \\
 &= \frac{16k+4}{4} \\
 \left\lfloor \frac{4k+1}{4} \right\rfloor &= m & \dots\dots (iii)
 \end{aligned}$$

Case (iv):

If $m \equiv 3 \pmod{4}$; let $m = 4k + 3$.

$$\begin{aligned}
 \gamma(G) &= 3 \left\lfloor \frac{m}{4} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor & [\because m \equiv 3 \pmod{4} \\
 &= 3 \left\lfloor \frac{4k+3}{4} \right\rfloor + \left\lfloor \frac{4k+3}{4} \right\rfloor & \left\lfloor \frac{4k+3}{4} \right\rfloor = \left\lfloor \frac{4k+3}{4} \right\rfloor \& \left\lfloor \frac{4k+3}{4} \right\rfloor = \left\lfloor \frac{4k}{4} \right\rfloor \\
 &= 3 \left\lfloor \frac{4k+3+1}{4} \right\rfloor + \left\lfloor \frac{4k}{4} \right\rfloor \\
 &= 3 \left(\frac{4k+4}{4} \right) + \frac{4k}{4} \\
 &= \frac{16k+12}{4} \\
 &= \frac{4(4k+3)}{4} = m & \dots\dots (iv)
 \end{aligned}$$

From equations (i), (ii), (iii) and (iv), we get

$$\gamma(G) = m.$$

Hence the proof.

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