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Properties Of Distributive, Neutral And Standard Elements Of A Lattice -

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Abstract: In this paper mainly we have obtained important properties of Distributive, Neutral and standard elements of a lattice. And also we have obtained certain equivalent conditions for a Distributive lattice.

Introduction: Distributive, standard element and Neutral elements are Defined in General Lattice Theory by Gabor Sazaz .In this paper first we gave important examples for these three elements. We also obtained important equivalent conditions for Distributive element which is obtained in Theorem-1. We also obtained important equivalent conditions for Standard element of a lattice which is obtained in Theorem-2. We also obtained important equivalent conditions for Neutral element of a lattice which is obtained in Theorem-3. It is to be observed that we have obtained important characteristics for Distributive ,Standard and Neutral element of a Lattice.

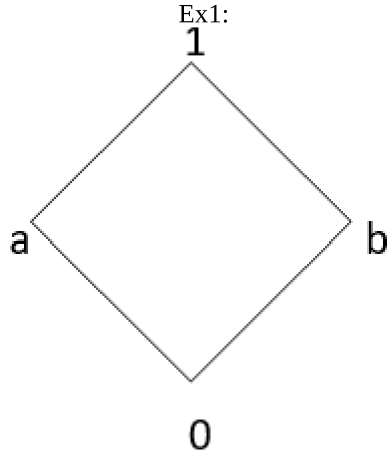
Key words: Distributive element of a lattice, Standard element of a lattice,
Neutral element of a lattice.

First we start with the following Definition:

Def1: Distributive element: Let ‘L’ be a lattice. An element $a \in L$ is called distributive element iff $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \ \forall \ x, y \in L$.

The following is an example which contains distributive elements

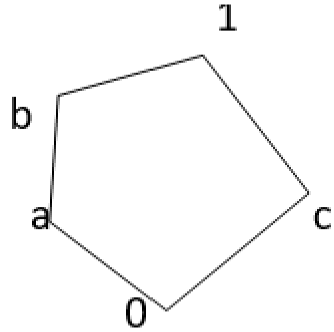
Ex1:



Clearly the elements {0,a,b,1} are distributive elements

NOTE1: For a Distributive Lattice, every element is an Example for a non-modular lattice which is non-distributive will contains distributive element

Ex 2:



Clearly the element b is distributive as $b \vee (a \wedge c) = b \vee 0 = b$

$(b \vee a) \wedge (b \vee c) = b \wedge (b \vee c) = b \wedge 1 = b$

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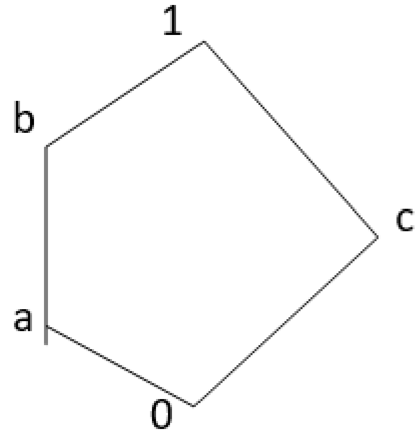
b V (b⋀c) =bV0=b imply that b⋀1=b. And the element a is not distributive element.

It is observed in the Lattice that Non-modular imply that Non-distributive.

Def2: An element 'a' of a lattice 'L' is called standard Iff
$x\wedge(a\vee y)=(x\wedge a)\vee(x\wedge y)$ for all $x,y\in L$

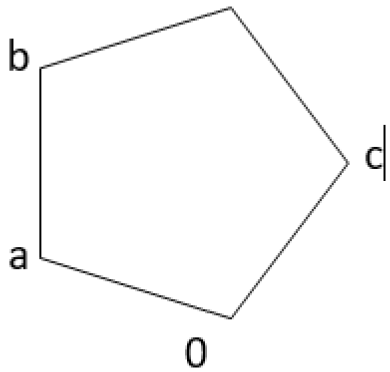
Ex 3: In N_5, 0,I,a are standard elements but 'c' is not standard element as



$b\wedge(c\vee a)=b\wedge 1=b$ and $(b\wedge c)\vee(b\wedge a)=0\vee (b\wedge a)=0\vee a=a$
But $a\neq b$ and hence $b\wedge(c\vee a)\neq (b\wedge c)\vee(b\wedge a)$

Def 3: An element 'a' of a lattice 'L' is called Neutral element iff
$(a \wedge x)\vee (x \wedge y)\vee (y \wedge a) =(a\vee x)\wedge (x \vee y)\wedge(y \vee a)$ for all $x,y\in L$

Ex 4: In N_5



0 and 1 are Neutral elements.
In M_3 0 and I are distributive lattice is distributive, standard and neutral.

REMARK1**:** Every element of a distributive Lattice is distributive, standard and neutral.
THEOREM 1: Let 'L' be a lattice and 'a' is an element of 'L' then the following conditions are equivalent.
1) 'a' is Distributive
2) The mapping φ :x⟶a∨x is a homomorphism on to [a)
3) The binary relation on $\equiv$ on L is defined by x≡y iff a∨x= a∨y is a congruence relation

1 ⇒ 2 Let 'a' be a distributive element then for for all x,y∈L a V(x⋀y)=(a V x) ⋀(a V y). Define φ : L on to (a] by φ (x)=a ∨x For any a, b $\in L,$ with a$\leq b$ so that a$\vee b = b$ and hence φ (b)= φ (a∨b) and hence the mapping φ is on to [a).Since for x, y$\in L,$ xφ V yφ =(a V x) ⋀(a V y) =a∨(x∨y) =(x∨y) φ . And also xφ ∧ yφ =(a V x) ⋀(a V y) = a∨(x∧y) =(x∧y) φ .

2⇒3. The binary relation φ is on to L is defined by x ≡y (φ_a)iff (a V x)= (a V y)is a congruence relation.

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Since a ≡a (φ_a) imply that φ isReflexive.

Let x ≡y (φ_a) iff x∨a=y∨a iff a∨x= a∨y

Iff y ≡x (φ_a) and hence φ_a is symmetric.

Let x ≡y (φ_a) and y ≡z (φ_a)Imply that (a V x)= (a∨y) and a∨y=a∨z imply that a∨x=a∨z so that x ≡z (φ_a) and hence is transitive.

Let x ≡y (φ_a)and y ≡p (φ_a)so that a∨x= a∨y and a∨y= a∨p
Imply that a∨x= a∨p
Now a∨(y∨p)=(a∨y)∨ (a∨p)=a∨(y∨p) imply that (x∨z,y∨p)∈ φ_a

And also a∨(x∧z)= (a∨x)∧ (a∨z)= (a∨y)∧ (a∨p)=a∨(y∧p) imply that (x∧z,y∧p)∈ φ_a a nd hence φ_a is a congruence relation.

3 => 1.Since x∨a =(a∨x)∨a⇒x≡a∨x(φ_a) and y≡a∨y(φ_a)

Imply that (x∧y)= (a∨x)∧ (a∨y) (φ_a)
a∨(x∧y)=a∨[(a∨x)∧ (a∨y)]= (a∨x)∧ (a∨y) and Hence 'a'is distributive.
THEOREM2: Let 'L' be a Lattice and 'a' is an element of 'L' then the following conditions are equivalent. x≡ y(φ_a)
1) 'a' is standard
2) The binary relation φ_a on L is defined by x≡ y(φ_a) iff (x∧y)∨a_1=x∨y for some $a_1 \le a$ is a congruence relation.
3) 'a' is distributive and for x,y∈L, a∧x=a∧y and a∨x=a∨y imply that x=y.
Proof: Suppose 1=> 2 holds and Let 'a' be a standard element

i.e., a∨(x∧y)= (a∨x)∧(a∨y). Define a binary relation φ_a on L by x≡ y(φ_a) iff (x∧y)∨a_1=x∨y for some $a_1 \le a$.

First it is to be observed that the relation φ_a is an equivalence relation.

Compatibility**:** Let x≤y and x≡ y(φ_a) i.e. x∨a_1=y with $a_1 \le a$

For t∈L, (x∨t)∨a_1=y∨t so that x∨t≡ y∨t (φ_a).Since y∧t ≤ y= x∨a_1≤ x∨a we have y∧t =(y∧t)∨(x∨a)= [(y∧t)∧x]∨ =[(y∧t)∧a]=(x∧t)∨a_2 where a_2= y∧t∧a≤a and hence φ_a is a Congruence relation.
2⇒3
Let φ_a be the congruence relation on a lattice 'L'
Let a∧x= a∧y and a∨x= a∨y
Since y≡ a∨ y(φ_a) , x∧y≡x∧(a∨x)≡ x(φ_a) and hence x= (x∧y)∨a_1 for a_1≤x imply that a_1≤a∧x= a∧y so that x=x∧y imply that x≤y. Similarly y=x∧y imply that y≤ x and hence x=y.
3=> 1 Let 'a' be a distributive element and for x, y∈L, a∧x=a∧y and a∨x=a∨y imply that x=y.
Define b=x∧(a∨y),c=(x∧a)∨(x∧y).
Now we claim that b=c.
By using 3 , it is sufficient to show that a∧b=a∧c and a∨b=a∨c .
Now a∨b=a∨[x∧(a∨y)]=(a∨x)∧(a∨y)=a∨(x∧a)∨(x∧y)=a∨c.
Similarly a∧b=a∧c imply that b=c.
THEOREM3: Let 'L' be the Lattice and let 'a' be an element of 'L' then the following conditions are equivalent.
1) Element 'a' is neutral
2) Element 'a' is distributive, 'a' is dually distributive and
a∧x= a∧y and a∨x= a∨y imply that x=y for all x,y∈L.
3) There is an embedding φ in to a direct product A×B where 'A' has <a,1> and B has <a,0> and aφ =<1,0>
4) For every x, y$\in L$, the sub lattice generated by a, x and y is Distributive.

Proof: 1=> 2.Let 'a' be neutral element then (a∧x)∨(x∧y)∨(y∧a)= (a∨x)∧(x∨y)∧(y∨a).For a≤x, a∨(x∧y)=(a∨x)∧(a∨y).Now, a∨[(a∧x)∨(x∧y)∨(y∧a)]= a∨[(a∨x)∧(x∨y)∧(y∨a)]=a∨[t∧(x∨y)] where t=(a∨x)∧ (y∨a)= (a∨t)∧[a∨(x∨y)]={a∨[(a∨x)∧(y∨a)]}∧a∨(x∨y) and hence 'a' is distributive .Similarly 'a' is Dually distributive and hence 1⇒2 holds.

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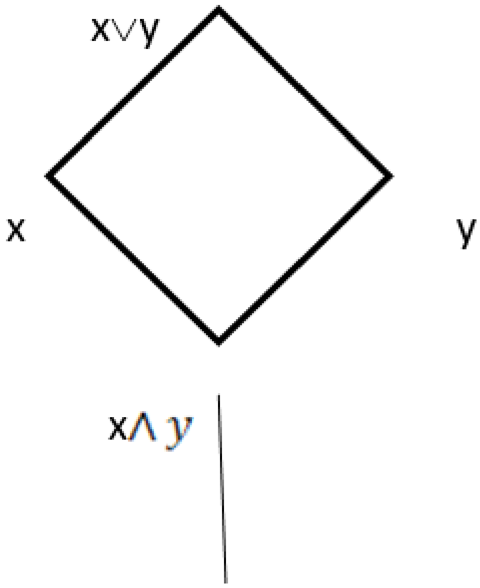
2⇒1 Suppose 'a' is distributive, and 'a' is dually distributive.Now we claim that 'a' is Neutral i.e,$a\vee(x\wedge y)=(a\vee x)\wedge(a\vee y)$

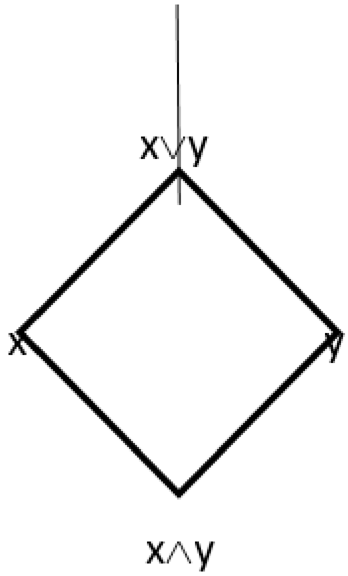
We have $(a\wedge x)\vee(x\wedge y)\vee(y\wedge a)=[(a\wedge x)\vee x]\wedge[(a\wedge x)\vee y\vee(y\wedge a)]=(a\vee x)\wedge(x\vee y)\wedge(y\vee a)$

Imply that 'a' is neutral. Hence 1 and 2 are equivalent.

It is easily observed that 2⇒3holds.

3⇒4 Case I: If x and y are comparable and if $x<y$ then $0<x<y$ imply that
$\{0,x,y\}$ is a sub lattice. Case II: if x and y are not comparable





Which are distributive
The sub lattice generated by
(a,x,y) in L_o is $P_1(a,x,y)$ where$x,y\in L_1$ $p_2(a,x,y)$ and $P_3(a,x,y)$
By using 3 There is an embedding φ in to a direct product $A\times B$ where 'A' has <a,1> and B has <a,0> and $a\varphi$ =<1,0>
Hence the sub lattice generated by a, x and y is Distributive

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