

Properties Of Distributive, Neutral And Standard Elements Of A Lattice -

Dr.K.V.R. Srinivas^{1*},

^{1*}Professor ,GIET Engineering College, Rajahmundry.

Abstract: In this paper mainly we have obtained important properties of Distributive, Neutral and standard elements of a lattice. And also we have obtained certain equivalent conditions for a Distributive lattice.

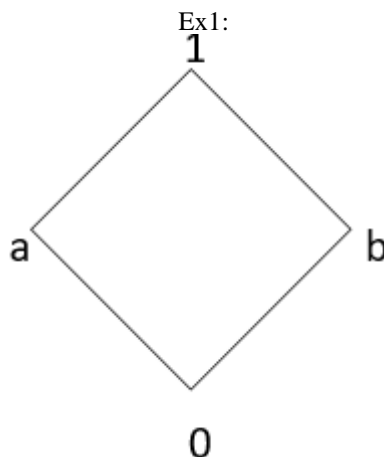
Introduction: Distributive, standard element and Neutral elements are Defined in General Lattice Theory by Gabor Szaz .In this paper first we gave important examples for these three elements. We also obtained important equivalent conditions for Distributive element which is obtained in Theorem-1. We also obtained important equivalent conditions for Standard element of a lattice which is obtained in Theorem-2. We also obtained important equivalent conditions for Neutral element of a lattice which is obtained in Theorem-3. It is to be observed that we have obtained important characteristics for Distributive ,Standard and Neutral element of a Lattice.

Key words: Distributive element of a lattice, Standard element of a lattice, Neutral element of a lattice.

First we start with the following Definition:

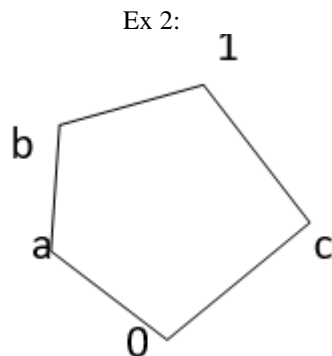
Def1: Distributive element: Let 'L' be a lattice. An element $a \in L$ is called distributive element iff $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \forall x, y \in L$.

The following is an example which contains distributive elements



Clearly the elements $\{0,a,b,1\}$ are distributive elements

NOTE1: For a Distributive Lattice, every element is an Example for a non-modular lattice which is non-distributive will contains distributive element



Clearly the element b is distributive as $b \vee (a \wedge c) = b \vee 0 = b$

$$(b \vee a) \wedge (b \vee c) = b \wedge (b \vee c) = b \wedge 1 = b$$

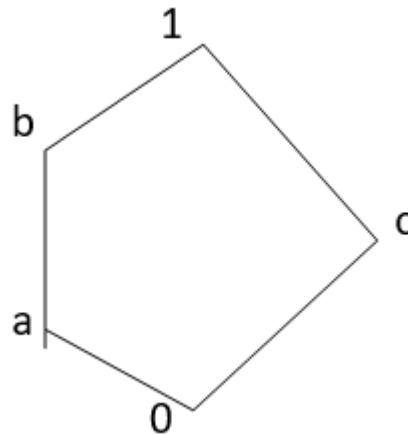
$b \vee (b \wedge c) = b \vee 0 = b$ imply that $b \wedge 1 = b$. And the element a is not distributive element.

It is observed in the Lattice that Non-modular imply that Non-distributive.

Def2: An element 'a' of a lattice 'L' is called standard iff

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y) \text{ for all } x, y \in L$$

Ex 3: In N_5 , 0, I, a are standard elements but 'c' is not standard element as



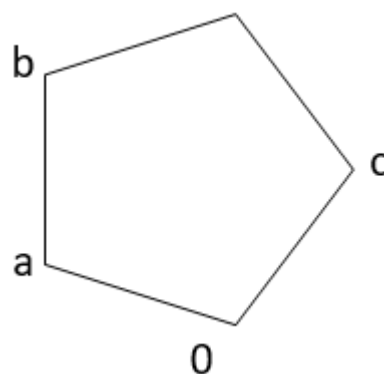
$$b \wedge (c \vee a) = b \wedge 1 = b \text{ and } (b \wedge c) \vee (b \wedge a) = 0 \vee (b \wedge a) = 0 \vee a = a$$

But $a \neq b$ and hence $b \wedge (c \vee a) \neq (b \wedge c) \vee (b \wedge a)$

Def 3: An element 'a' of a lattice 'L' is called Neutral element iff

$$(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a) \text{ for all } x, y \in L$$

Ex 4: In N_5



0 and 1 are Neutral elements.

In M_3 0 and I are distributive lattice is distributive, standard and neutral.

REMARK1: Every element of a distributive Lattice is distributive, standard and neutral.

THEOREM 1: Let 'L' be a lattice and 'a' is an element of 'L' then the following conditions are equivalent.

1) 'a' is Distributive

2) The mapping $\varphi : x \rightarrow a \vee x$ is a homomorphism on to [a]

3) The binary relation \equiv on L is defined by $x \equiv y$ iff $a \vee x = a \vee y$ is a congruence relation

$1 \Rightarrow 2$ Let 'a' be a distributive element then for all $x, y \in L$ $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$. Define $\varphi : L$ on to [a] by $\varphi(x) = a \vee x$ For any $a, b \in L$, with $a \leq b$ so that $a \vee b = b$ and hence $\varphi(b) = \varphi(a \vee b)$ and hence the mapping φ is on to [a]. Since for $x, y \in L$, $x \varphi \vee y \varphi = (a \vee x) \wedge (a \vee y) = a \vee (x \vee y) = (x \vee y) \varphi$. And also $x \varphi \wedge y \varphi = (a \vee x) \wedge (a \vee y) = a \vee (x \wedge y) = (x \wedge y) \varphi$.

$2 \Rightarrow 3$. The binary relation φ is on to L is defined by $x \equiv y$ (φ a) iff $(a \vee x) = (a \vee y)$ is a congruence relation.

Since $a \equiv a (\varphi_a)$ imply that φ is Reflexive.

Let $x \equiv y (\varphi_a)$ iff $x \vee a = y \vee a$ iff $a \vee x = a \vee y$

Iff $y \equiv x (\varphi_a)$ and hence φ_a is symmetric.

Let $x \equiv y (\varphi_a)$ and $y \equiv z (\varphi_a)$ imply that $(a \vee x) = (a \vee y)$ and $a \vee y = a \vee z$ imply that $a \vee x = a \vee z$ so that $x \equiv z (\varphi_a)$ and hence is transitive.

Let $x \equiv y (\varphi_a)$ and $y \equiv p (\varphi_a)$ so that $a \vee x = a \vee y$ and $a \vee y = a \vee p$

Imply that $a \vee x = a \vee p$

Now $a \vee (x \vee p) = (a \vee x) \vee (a \vee p) = a \vee (y \vee p)$ imply that $(x \vee z, y \vee p) \in \varphi_a$

And also $a \vee (x \wedge z) = (a \vee x) \wedge (a \vee z) = (a \vee y) \wedge (a \vee p) = a \vee (y \wedge p)$ imply that $(x \wedge z, y \wedge p) \in \varphi_a$ and hence φ_a is a congruence relation.

$3 \Rightarrow 1$. Since $x \vee a = (a \vee x) \vee a \Rightarrow x \equiv a \vee x (\varphi_a)$ and $y \equiv a \vee y (\varphi_a)$

Imply that $(x \wedge y) = (a \vee x) \wedge (a \vee y) (\varphi_a)$

$a \vee (x \wedge y) = a \vee [(a \vee x) \wedge (a \vee y)] = (a \vee x) \wedge (a \vee y)$ and Hence 'a' is distributive.

THEOREM2: Let 'L' be a Lattice and 'a' is an element of 'L' then the following conditions are equivalent. $x \equiv y (\varphi_a)$

1) 'a' is standard

2) The binary relation φ_a on L is defined by $x \equiv y (\varphi_a)$ iff $(x \wedge y) \vee a_1 = x \vee y$ for some $a_1 \leq a$ is a congruence relation.

3) 'a' is distributive and for $x, y \in L$, $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ imply that $x = y$.

Proof: Suppose $1 \Rightarrow 2$ holds and Let 'a' be a standard element

i.e., $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$. Define a binary relation φ_a on L by $x \equiv y (\varphi_a)$ iff $(x \wedge y) \vee a_1 = x \vee y$ for some $a_1 \leq a$.

First it is to be observed that the relation φ_a is an equivalence relation.

Compatibility: Let $x \leq y$ and $x \equiv y (\varphi_a)$ i.e. $x \vee a_1 = y$ with $a_1 \leq a$

For $t \in L$, $(x \vee t) \vee a_1 = y \vee t$ so that $x \vee t \equiv y \vee t (\varphi_a)$. Since $y \wedge t \leq y = x \vee a_1 \leq x \vee a$ we have $y \wedge t = (y \wedge t) \vee (x \vee a) = [(y \wedge t) \wedge x] \vee [(y \wedge t) \wedge a] = (x \wedge t) \vee a_2$ where $a_2 = y \wedge t \wedge a \leq a$ and hence φ_a is a Congruence relation.

$2 \Rightarrow 3$

Let φ_a be the congruence relation on a lattice 'L'

Let $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$

Since $y \equiv a \vee y (\varphi_a)$, $x \wedge y \equiv x \wedge (a \vee y) \equiv x (\varphi_a)$ and hence $x = (x \wedge y) \vee a_1$ for $a_1 \leq x$ imply that $a_1 \leq a \wedge x = a \wedge y$ so that $x = x \wedge y$ imply that $x \leq y$. Similarly $y = x \wedge y$ imply that $y \leq x$ and hence $x = y$.

$3 \Rightarrow 1$ Let 'a' be a distributive element and for $x, y \in L$, $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ imply that $x = y$.

Define $b = x \wedge (a \vee y)$, $c = (x \wedge a) \vee (x \wedge y)$.

Now we claim that $b = c$.

By using 3, it is sufficient to show that $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$.

Now $a \vee b = a \vee [x \wedge (a \vee y)] = (a \vee x) \wedge (a \vee y) = a \vee (x \wedge a) \vee (x \wedge y) = a \vee c$.

Similarly $a \wedge b = a \wedge c$ imply that $b = c$.

THEOREM3: Let 'L' be the Lattice and let 'a' be an element of 'L' then the following conditions are equivalent.

1) Element 'a' is neutral

2) Element 'a' is distributive, 'a' is dually distributive and

$a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ imply that $x = y$ for all $x, y \in L$.

3) There is an embedding φ in to a direct product $A \times B$ where 'A' has $\langle a, 1 \rangle$ and B has $\langle a, 0 \rangle$ and $a \varphi = \langle 1, 0 \rangle$

4) For every $x, y \in L$, the sub lattice generated by a, x and y is Distributive.

Proof: $1 \Rightarrow 2$. Let 'a' be neutral element then $(a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$. For $a \leq x$, $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$. Now, $a \vee [(a \wedge x) \vee (x \wedge y) \vee (y \wedge a)] = a \vee [(a \vee x) \wedge (x \vee y) \wedge (y \vee a)] = a \vee [t \wedge (x \vee y)]$ where $t = (a \vee x) \wedge (y \vee a) = (a \vee t) \wedge [a \vee (x \vee y)] = \{a \vee [(a \vee x) \wedge (y \vee a)]\} \wedge a \vee (x \vee y)$ and hence 'a' is distributive. Similarly 'a' is Dually distributive and hence $1 \Rightarrow 2$ holds.

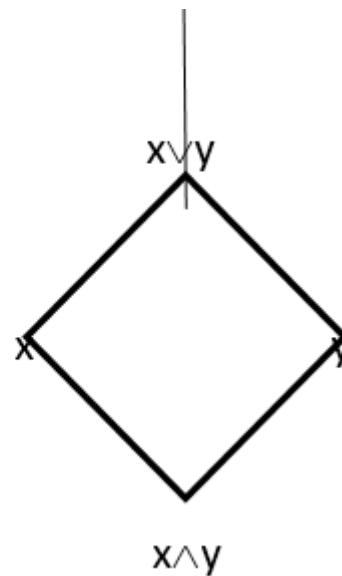
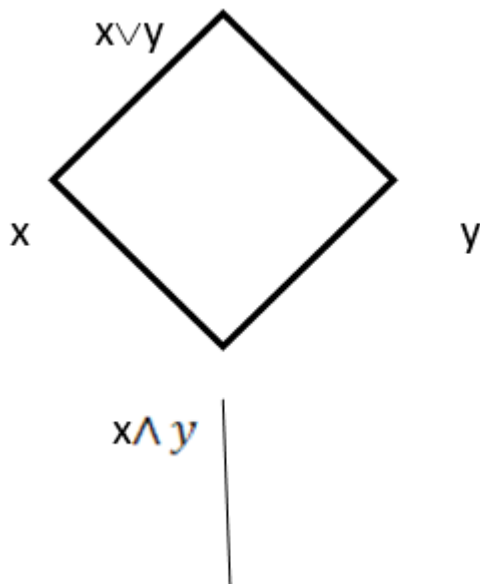
$2 \Rightarrow 1$ Suppose 'a' is distributive, and 'a' is dually distributive. Now we claim that 'a' is Neutral i.e., $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$

$$\text{We have } (a \wedge x) \vee (x \wedge y) \vee (y \wedge a) = [(a \wedge x) \vee x] \wedge [(a \wedge x) \vee y \vee (y \wedge a)] = (a \vee x) \wedge (x \vee y) \wedge (y \vee a)$$

Imply that 'a' is neutral. Hence 1 and 2 are equivalent.

It is easily observed that $2 \Rightarrow 3$ holds.

$3 \Rightarrow 4$ Case I: If x and y are comparable and if $x < y$ then $0 < x < y$ imply that $\{0, x, y\}$ is a sub lattice. Case II: if x and y are not comparable



Which are distributive

The sub lattice generated by

(a, x, y) in L_0 is $P_1(a, x, y)$ where $x, y \in L_1$ $p_2(a, x, y)$ and $P_3(a, x, y)$

By using 3 There is an embedding ϕ in to a direct product $A \times B$ where 'A' has $\langle a, 1 \rangle$ and B has $\langle a, 0 \rangle$ and a $\phi = \langle 1, 0 \rangle$

Hence the sub lattice generated by a, x and y is Distributive

References:

1. A Text book of General Lattice Theorey : Birkhoff